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Concrete construction and properties of the difference equation derived from the cellular automaton using the filtration technique

Tomonori Watanabe

Graduate School of Science & Technology, Chiba University, Chiba, 263-8522, Japan

E-mail: nabe@meneth.tm.chiba-u.ac.jp

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Abstract

Following the proposal of a filtration technique by Nobe, Satsuma and Tokihiro, we concretely construct partial difference equations, which preserve any time evolution patterns of cellular automaton (CA) stably by the filtration technique. We illustrate how to develop a method of filtration for applying to the typical two spatial dimensional CA rule—the game of life—and verify that the filtration method provides the stable difference equation associated with the CA, compared with the inverse ultradiscretization. Besides, in order to discuss whether the filtration technique can lead one to partial differential equations from CA rules, we show a derivation of the Burgers equation from Rule 184 CA via the discrete Burgers equation constructed by the filtration method as an example.

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1. Introduction

Various complicated patterns can be seen in physical phenomena. In many cases, they are modelled and analysed by cellular automaton (CA), because it is a discrete system which can produce complex patterns in spite of its simplicity. However, recently, the complicated patterns have been discussed in view of the continuous dynamical systems. For example, a regular self-similar structure like a Sierpinski gasket is shown in the excitable reaction diffusion systems by computer simulations [1, 2]. Therefore, it would be significant to understand how the discrete systems and the continuous systems relate. Actually, CA associated with the continuous system is discussed widely. For instance, the problem of CA is proposed as Wolfram's 9th problem, namely, '*what is the correspondence between cellular automata and continuous systems?*' [3], and rather for physical interest, a CA model of excitable media including curvature and dispersion is investigated [4]. (More general topics are collected in [5].)

For integrable CA, a direct link between CAs and integrable partial differential equations is successfully given by ultradiscretization, which is a procedure for transforming a difference equation to a CA [6–8]. In the procedure, by taking a limit of a parameter which exists in the solutions and the equation, we obtain the analytical expression for the time evolution patterns of the CA. The patterns obtained preserve the features of the original solutions naturally. A typical example is the soliton cellular automaton [9]. The ultradiscretization is not specific to soliton systems and can be applied widely. Indeed, there exists an example where the ultradiscrete limit is applied to a chaotic equation [10].

On the other hand, the inverse process of the ultradiscretization—inverse ultradiscretization—which is a procedure for transforming a CA to a difference equation, is also considered in order to examine the connection between CA and continuous systems. However, the *naive* approach of the inverse ultradiscretization has a serious difficulty in constructing the difference equation which stably preserves the original patterns of the CA [11]. Recently, Nobe, Satsuma and Tokihiro proposed a universal method of inverse ultradiscretization which stably preserves *any* time evolution pattern of *any* CA [12]. In their paper, they have defined a stable piecewise linear map associated with a given CA and discussed its importance in inverse ultradiscretization. They have presented a general method to construct a stable piecewise linear map from a CA, and a general formalism to construct a stable map by introducing the notion of *filtration*.

In this paper, referring to the idea of *filtration*, we concretely construct the partial difference equations which preserve any time evolution patterns of the CA stably. In this study, we focus on two points as follows. First, we investigate how to develop a method of filtration for applying to the concrete and complicated system, and show how to obtain the stable difference equation uniquely associated with the CA. Secondly, we examine whether the filtration technique can lead one to partial differential equations from CA rules. For the first point, we concretely derive the difference equation corresponding to the two spatial dimensional CA rule—the game of life—as a typical example, and verify that the stable complex patterns are produced by the constructed difference equation, compared with the inverse ultradiscretization. (We should comment on the main reasons why the game of life is selected as follows. It is well known that the game of life is one of the typical two spatial dimensional CA. It simulates the dynamical evolution of a society of living organisms by means of a simple algorithm. Despite its simplicity, it provides the complex dynamics of the game. Various approaches are adopted to understand the dynamics (for example, see [13] and references therein).) For the second point, since Nishinari and Takahashi concluded that the relation between the Burgers equation and one of the elementary CA (ECA)—Rule 184 CA—was clarified via the discrete Burgers equation and the ultradiscrete Burgers equation derived from the discrete Burgers equation with ultradiscretizaton [14], referring to this result, we derive the Burgers equation from Rule 184 CA through the filtration technique. We discuss whether the filtration technique is a method which can contribute to revealing the relation between the continuous dynamical system and the CA rule.

This paper is organized as follows: In section 2, we review the naive approach to constructing a difference equation by inverse ultradiscretization and weigh the difficulty in obtaining a difference equation which stably exhibits any time evolution patterns of the game of life. In section 3, we construct a partial difference equation which preserves any time evolution patterns of the game of life by introducing the filtration functions. In section 4, we apply the procedure obtained in the previous section to Rule 184 CA in order to examine the connection between CA and continuous systems. Section 5 is devoted to concluding remarks.

2. The naive approach by inverse ultradiscretization

In this section, we consider the naive approach to constructing the partial difference equation associated with the game of life by means of inverse ultradiscretization. It is usually done with the following steps:

- We rewrite the time evolution rule of a CA as a piecewise linear equation with max and plus algebra.
- By introducing a parameter ε, we make a tropical variable change, that is, we replace $\max[a, b]$ by $\varepsilon \log[\mathrm{e}^{a/\varepsilon} + \mathrm{e}^{b/\varepsilon}]$.

The time evolution rule of the game of life is given as follows. Let the site on the two spatial dimensional lattice have the value 0 or 1. The site whose value is 1 is called the *ON* site. When the value of the site is 0, it is called the *OFF* site. The value of the site at the next timestep is determined by the values of its site and the nearest neighbour sites (Moore neighbourhoods) at the present timestep. Two cases are allowed to make the site be *ON* at the next timestep. The first case is that the site is *ON* and two or three *ON* sites exist as its nearest neighbours. The second one is that the site is *OFF* and three *ON* sites exist as its nearest neighbours. In other cases, the site keeps or becomes *OFF* at the next timestep. Generally the time evolution rule of the game of life is denoted by

$$v_{i,j}^{t+1} = F_{LG}\left(v_{i-1,j-1}^t, v_{i,j-1}^t, v_{i+1,j-1}^t, v_{i-1,j}^t, v_{i,j}^t, v_{i+1,j}^t, v_{i-1,j+1}^t, v_{i,j+1}^t, v_{i+1,j+1}^t\right) \quad (1)$$

where $v_{i,j}^t \in \{0, 1\}$ is the value of the $\{i, j\}$ site ($i, j \in \mathbb{Z}$) on the two spatial dimensional lattice at timestep $t \in \mathbb{Z}$, and F_{LG} is a map from $\underbrace{\{0, 1\} \times \{0, 1\} \times \cdots \times \{0, 1\}}_{9 \text{ times}}$ to $\{0, 1\}$ and represents the above rule.

In order to rewrite the time evolution rule of the game of life as a piecewise linear equation, we give the three functions A, B and C as follows.

$$A = \begin{cases} 1 & \text{(three sites are } ON \text{ in the nearest neighbours)} \\ 0 & \text{(otherwise)} \end{cases} \quad (2)$$

$$B = \begin{cases} 1 & \text{(two sites are } ON \text{ in the nearest neighbours)} \\ 0 & \text{(otherwise)} \end{cases} \quad (3)$$

$$C = \begin{cases} 1 & \text{(three sites are } ON \text{ in its own and the nearest neighbours)} \\ 0 & \text{(otherwise)} \end{cases}. \quad (4)$$

Then, we obtain the piecewise linear equation with max and plus algebra which shows that the value of a certain site on a two-dimensional lattice at timestep $t + 1$ ($u^{t+1} \in \mathbb{R}$) is determined by the values of its own site and the nearest neighbour sites at the previous timestep t $\left(u^t \text{ and } u_k^t\ (k = 1, 2, \ldots, 8) \in \mathbb{R}\right)$. We consider the condition that a piecewise linear equation has the symmetry of the space and inversion between *ON* and *OFF*, and the scaling invariability of the dependent variable (the equation is unchanged when $u^t \mapsto \kappa u^t$, $\kappa \in \mathbb{R}$). Thus, one of the simplest piecewise linear equations which obeys the time evolution rule of the game of life is given by

$$u^{t+1} = u^t + A + \max(A + B, B + C, u^t + B, u^t + C) - \max(u^t, A) - \max(u^t, B). \quad (5)$$

Here, we can get the functions A, B and C explicitly by using the functions Lij and $LCij$. The functions Lij and $LCij$ are defined by

$$Lij := \max\left(u_{a_1}^t + u_{a_2}^t + \cdots + u_{a_i}^t - u_{b_1}^t - u_{b_2}^t - \cdots - u_{b_j}^t\right) \quad (6)$$

$$LCij := \max\left(u_{a_1'}^t + u_{a_2'}^t + \cdots + u_{a_i'}^t - u_{b_1'}^t - u_{b_2'}^t - \cdots - u_{b_j'}^t\right) \quad (7)$$

where $u^t_{a_l}$ and $u^t_{b_m}$ $(a_1 \leqslant a_l \leqslant a_i, b_1 \leqslant b_m \leqslant b_j)$ are assigned from u^t_k $(k = 1, 2, \ldots, 8)$ as mutually different. In the case of $LCij$, $u^t_{a'_l}$ and $u^t_{b'_m}$ $(a'_1 \leqslant a'_l \leqslant a'_i, b'_1 \leqslant b'_m \leqslant b'_j)$ are assigned from u^t and u^t_k $(k = 1, 2, \ldots, 8)$ as mutually different. For example,

$$L12 = \max\left(u^t_{a_1} - u^t_{b_1} - u^t_{b_2}\right) \tag{8}$$

$$= \max\left(u^t_1 - u^t_2 - u^t_3, u^t_1 - u^t_2 - u^t_4, u^t_1 - u^t_2 - u^t_5, \ldots\right) \tag{9}$$

$$LC21 = \max\left(u^t_{a'_1} + u^t_{a'_2} - u^t_{b'_1}\right) \tag{10}$$

$$= \max\left(u^t + u^t_1 - u^t_2, u^t + u^t_1 - u^t_3, u^t + u^t_1 - u^t_4, \ldots\right). \tag{11}$$

When we calculate the functions Lij and $LCij$, the values of Lij and $LCij$ are given by tables 1 and 2, respectively. Moreover, the functions A, B and C are shown as a combination of Lij and $LCij$. Table 3 shows the expressions of the functions A, B and C by the combination of Lij and $LCij$.

From equation (5), we obtain the partial difference equation by making a tropical variable exchange (inverse ultradiscretization) as

$$u^{t+1} = u^t + A + \varepsilon \log\left[\exp\left(\frac{A+B}{\varepsilon}\right) + \exp\left(\frac{B+C}{\varepsilon}\right) + \exp\left(\frac{u^t+B}{\varepsilon}\right) + \exp\left(\frac{u^t+C}{\varepsilon}\right)\right] - \varepsilon \log\left[\exp\left(\frac{u^t}{\varepsilon}\right) + \exp\left(\frac{A}{\varepsilon}\right)\right] - \varepsilon \log\left[\exp\left(\frac{u^t}{\varepsilon}\right) + \exp\left(\frac{B}{\varepsilon}\right)\right] \tag{12}$$

Table 1. The values of Lij against the number of ON sites in the nearest neighbours.

	0	1	2	3	4	5	6	7	8
$L11$	0	1	1	1	1	1	1	1	0
$L12$	0	1	1	1	1	1	1	0	−1
$L21$	0	1	2	2	2	2	2	2	1
$L22$	0	1	2	2	2	2	2	1	0
$L23$	0	1	2	2	2	2	1	0	−1
$L32$	0	1	2	3	3	3	3	2	1
$L33$	0	1	2	3	3	3	2	1	0
$L34$	0	1	2	3	3	2	1	0	−1
$L43$	0	1	2	3	4	4	3	2	1
$L44$	0	1	2	3	4	3	2	1	0

Table 2. The values of $LCij$ against the number of ON sites in its own and the nearest neighbours.

	0	1	2	3	4	5	6	7	8	9
$LC11$	0	1	1	1	1	1	1	1	1	0
$LC12$	0	1	1	1	1	1	1	1	0	−1
$LC21$	0	1	2	2	2	2	2	2	2	1
$LC22$	0	1	2	2	2	2	2	2	1	0
$LC23$	0	1	2	2	2	2	2	1	0	−1
$LC32$	0	1	2	3	3	3	3	3	2	1
$LC33$	0	1	2	3	3	3	3	2	1	0
$LC34$	0	1	2	3	3	3	2	1	0	−1
$LC43$	0	1	2	3	4	4	4	3	2	1
$LC44$	0	1	2	3	4	4	3	2	1	0

Table 3. The expressions of the functions A, B and C by the combination of Lij and $LCij$.

	A
1	$-L22 + L32 + L34 - L44$
2	$L11 - L12 - L21 + L32 + L34 - L44$
3	$-L11 + L12 + L21 - L22 - L23 + L33 + L34 - L44$
4	$-L23 + L33 + L34 - L44$
5	$L11 - L12 - L21 + L22 - L23 + L33 + L34 - L44$
6	$-L22 + L32 + L33 - L43$
7	$L11 - L12 - L21 + L32 + L33 - L43$
	B
1	$-L11 + L21 + L22 - L32 - L33 + L34 + L43 - L44$
2	$-L11 + L21 + L23 - L33$
3	$-L12 + L22 + L23 - L33$
4	$-L11 + L21 + L22 - L32$
5	$-L11 + L21 + L23 - L34 - L43 + L44$
6	$-L12 + L22 + L23 - L34 - L43 + L44$
7	$-L11 + L21 + L22 - L32 + L33 - L34 - L43 + L44$
	C
1	$-LC22 + LC32 + LC34 - LC44$
2	$LC11 - LC12 - LC21 + LC32 + LC34 - LC44$
3	$-LC11 + LC12 + LC21 - LC22 - LC23 + LC33 + LC34 - LC44$
4	$-LC23 + LC33 + LC34 - LC44$
5	$LC11 - LC12 - LC21 + LC22 - LC23 + LC33 + LC34 - LC44$
6	$-LC22 + LC32 + LC33 - LC43$
7	$LC11 - LC12 - LC21 + LC32 + LC33 - LC43$

or replacing u^t/ε, $\exp(A/\varepsilon)$, $\exp(B/\varepsilon)$ and $\exp(C/\varepsilon)$ with U^t, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, respectively.

$$U^{t+1} = U^t + \log\left[\tilde{A}\left(\frac{C_1\tilde{C}}{C_2\exp(U^t) + \tilde{A}} + \frac{C_3\tilde{B}}{C_4\exp(U^t) + \tilde{B}}\right)\right] \tag{13}$$

where C_1, C_2, C_3 and C_4 are the positive constants which vanish in the ultradiscrete limit ($\varepsilon \to +0$) so that $U^t \equiv 0$ (for $\forall t$ and any sites) remains a solution in the inverse ultradiscretized process. The functions $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in the above equation are explicitly shown by the inverse ultradiscretization of Lij and $LCij$. We define the functions Xij and $XCij$ by

$$Xij := \sum_{a_i,b_j} \mathrm{e}^{\left(U^t_{a_1}+U^t_{a_2}+\cdots+U^t_{a_i}-U^t_{b_1}-U^t_{b_2}-\cdots-U^t_{b_j}\right)} \tag{14}$$

$$XCij := {\sum_{a_i,b_j}}' \mathrm{e}^{\left(U^t_{a_1}+U^t_{a_2}+\cdots+U^t_{a_i}-U^t_{b_1}-U^t_{b_2}-\cdots-U^t_{b_j}\right)} \tag{15}$$

where the summation $\sum_{a_i,b_j}$ means the sum of all the cases that $U^t_{a_l}$ and $U^t_{b_m}(a_1 \leqslant a_l \leqslant a_i, b_1 \leqslant b_m \leqslant b_j)$ are assigned from U^t_k $(k = 1, 2, \ldots, 8)$ as mutually different, and the summation $\sum'_{a_i,b_j}$ is the sum of all the cases $U^t_{a_l}$ and $U^t_{b_m}$ $(a_1 \leqslant a_l \leqslant a_i, b_1 \leqslant b_m \leqslant b_j)$ are assigned from U^t and U^t_k $(k = 1, 2, \ldots, 8)$ as mutually different. Then the functions Lij and $LCij$ are replaced by

$$Lij = \varepsilon \log Xij \qquad (\varepsilon \to +0) \tag{16}$$

$$LCij = \varepsilon \log XCij \qquad (\varepsilon \to +0). \tag{17}$$

For simplicity, we consider one case of table 3 as an example. (This process is similarly applied to the other cases.) When the function is given by

$$A = -L22 + L32 + L33 - L43 \tag{18}$$

$$\simeq -\varepsilon \log X22 + \varepsilon \log X32 + \varepsilon \log X33 - \varepsilon \log X43 \tag{19}$$

we obtain the function $\tilde{A}$ as

$$\tilde{A} = \exp(A/\varepsilon) = \frac{X32\, X33}{X22\, X43}. \tag{20}$$

Therefore, when the functions B and C are given by

$$B = -L12 + L22 + L23 - L33 \tag{21}$$

$$C = -LC22 + LC32 + LC33 - LC43 \tag{22}$$

from the equation (13), we have the difference equation

$$U^{t+1} = U^t + \log\left[\frac{X32\, X33}{X22\, X43}\left(\frac{\frac{231}{560}\frac{XC32\, XC33}{XC22\, XC43}}{\exp(U^t) + \frac{X32\, X33}{X22\, X43}} + \frac{\frac{7}{40}\frac{X22\, X23}{X12\, X33}}{\exp(U^t) + \frac{X22\, X23}{X12\, X33}}\right)\right]. \tag{23}$$

In this equation, the constants C_1, C_2, C_3 and C_4 are determined to satisfy the condition that the equation has the trivial solution $U^t \equiv 0$ for $\forall t$ and any sites. Figure 1 shows the results of simulating the equation (23). In this figure, the parameter ε is 0.1. The white site means the value of it is 10.0 $\left(u^t_{i,j} = 1.0\right)$. The black site is the value 0.0 $\left(u^t_{i,j} = 0.0\right)$. The grey level corresponds to the magnitude of the value from zero to ten. This figure shows the typical time evolution pattern of the game of life—a blinker—is gradually dispersing and vanishing. (Note that the original blinker pattern of the game of life keeps its values and two types of the patterns evolve by turns, see figure 3.) This implies the difference equation (23) does not have a solution which preserves the blinker pattern as long as the parameter ε is finite. This is the difficulty in constructing a difference equation from a CA by the naive approach.

3. Difference equation associated with the game of life

We define two monotonic and smooth real functions $X_0(x)$ and $X_1(x)$ which keep almost constant (0 or 1) around $x = 0$ and $x = 1$. The configurations of these functions (the filtration functions) are shown in figure 2. As an example, the functions are explicitly given by

$$X_0(x) = \frac{1}{2}\left(1 + \tanh\left(\frac{1-2x}{2\varepsilon}\right)\right) \tag{24}$$

$$X_1(x) = \frac{1}{2}\left(1 - \tanh\left(\frac{1-2x}{2\varepsilon}\right)\right) \tag{25}$$

where ε is a small constant. The point of these functions is one condition:

> the filtration functions behave as $X_0(0) = 1$, $X_0(1) = 0$, $X_1(0) = 0$ and $X_1(1) = 1$ in the limit of $\varepsilon \to +0$. (26)

This behaviour works as the *filtration* for stabilizing. In fact, by using the functions $X_0(x)$ and $X_1(x)$, we can automatically construct the difference equation which produces the time evolution pattern of the CA stably. The rule of CA which depends on the values of the sites at the prior timestep is represented by the linear combination of $X_0(x)$ and $X_1(x)$. (Note that, therefore, the filtration functions are not always (24) and (25). They have to satisfy only the condition (26). Indeed, the other filtration functions will be applied in the next section.)

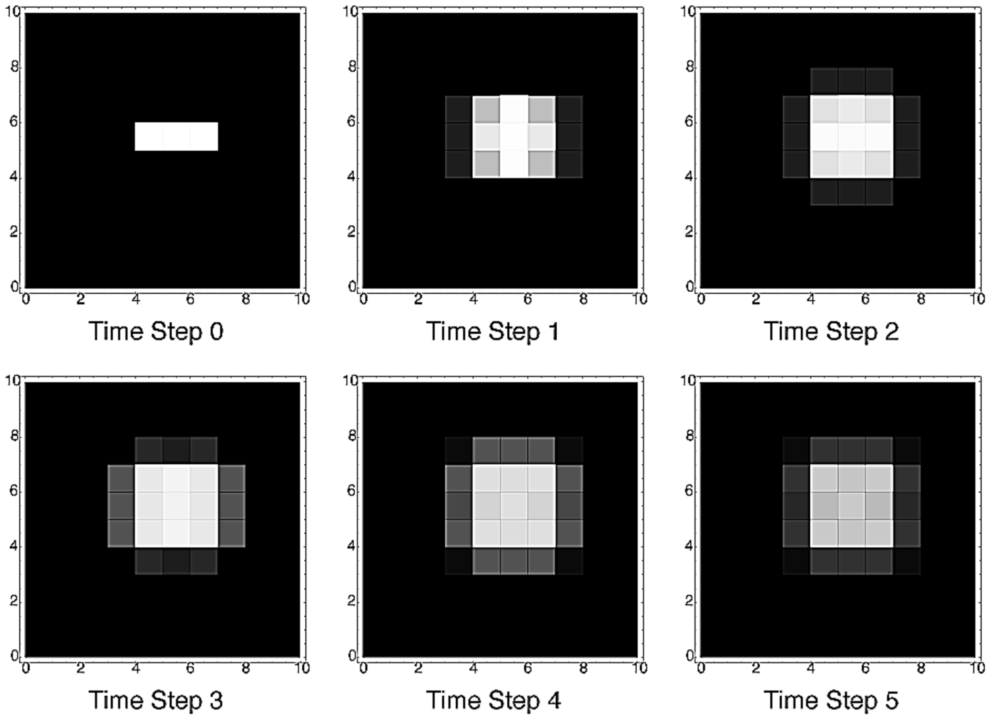

Figure 1. Blinker (the inverse ultradiscretization). The numerical result of the difference equation (23). The parameter ε is 0.1. A white site means the value of it is 10.0 $\left(u^t_{i,j} = 1.0\right)$. A black site has the value 0.0 $\left(u^t_{i,j} = 0.0\right)$. The grey levels correspond to the magnitude of the value from zero to ten.

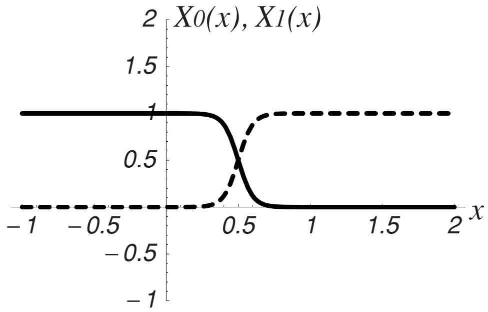

Figure 2. The configuration of the filtration functions (X_0: solid line, X_1: broken line).

In the case of the game of life, the partial difference equation is given by the linear combination of $X_0(x)$ and $X_1(x)$:

$$\begin{aligned} U^{t+1}_{i,j} &= K_F\left(U^t_{i-1,j-1}, U^t_{i-1,j}, U^t_{i-1,j+1}, U^t_{i,j-1}, U^t_{i,j}, U^t_{i,j+1}, U^t_{i+1,j-1}, U^t_{i+1,j}, U^t_{i+1,j+1}\right) \\ &= \sum_{i_0=0}^{1}\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\sum_{i_4=0}^{1}\sum_{i_5=0}^{1}\sum_{i_6=0}^{1}\sum_{i_7=0}^{1}\sum_{i_8=0}^{1} F_{LG}(i_0, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8) \\ &\quad \times X_{i_1}\left(U^t_{i-1,j-1}\right) X_{i_2}\left(U^t_{i-1,j}\right) X_{i_3}\left(U^t_{i-1,j+1}\right) X_{i_4}\left(U^t_{i,j-1}\right) X_{i_5}\left(U^t_{i,j}\right) X_{i_6}\left(U^t_{i,j+1}\right) \\ &\quad \times X_{i_7}\left(U^t_{i+1,j-1}\right) X_{i_8}\left(U^t_{i+1,j}\right) X_{i_9}\left(U^t_{i+1,j+1}\right) \end{aligned} \tag{27}$$

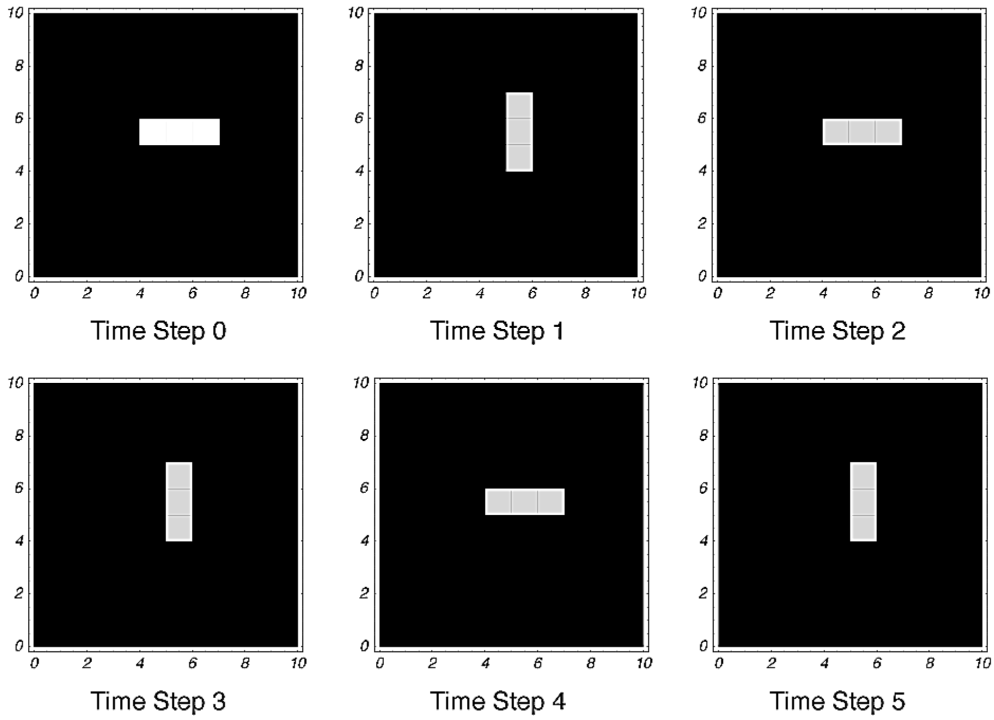

Figure 3. Blinker (by the filtration technique). The numerical result of the difference equation (27) with the filtration functions (24) and (25) The parameter ε is 0.1. A white site means the value of it is 10.0 $\left(U_{i,j}^t = 10.0\right)$. A black site has the value 0.0 $\left(U_{i,j}^t = 0.0\right)$. The grey levels correspond to the magnitude of the value from zero to ten. The initial conditions and each timestep of the simulation correspond to figure 1. In fact, only two colours (black and grey) appear after timestep 1 and the value of the grey sites means unity.

where $U_{i,j}^t \in \mathbb{R}$ is the value at time $t \in \mathbb{Z}$ on the $\{i, j\}$ site $(i, j \in \mathbb{Z})$ and K_F is a map from $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{9\ \text{times}}$ to $\mathbb{R}$, and F_{LG} is the map which represents the time evolution rule of the game of life. Here, we define the map K_F as *stable* [12] when there exists a positive number $\delta\left(0 < \delta < \frac{1}{2}\right)$ such that if $\forall i, \forall j$, $\left|U_{i,j}^t - v_{i,j}^t\right| < \delta$ then

$$\left|K_F\left(U_{i-1,j-1}^t, \ldots, U_{i+1,j+1}^t\right) - F_{LG}\left(v_{i-1,j-1}^t, \ldots, v_{i+1,j+1}^t\right)\right| < \delta.$$

Then we find that equation (27) preserves *any* time evolution pattern of the CA in the sense that, when the initial values of the equation (27) approximately take values in $\{0, 1\}$ with a tolerance of δ, the time evolution pattern of the equation (27) coincides with that of (1) if we round off the dependent variables in the pattern.

Figure 3 shows the results of simulating the difference equation (27) with the filtration functions $X_0(x)$ and $X_1(x)$ which are given by (24) and (25) with $\varepsilon = 0.1$. In figure 3, a white site means the value of it is 10.0 $\left(U_{i,j}^t = 10.0\right)$. A black site has the value 0.0 $(U_{i,j}^t = 0.0)$. The grey levels correspond to the magnitude of the value from zero to ten. The initial conditions and each timestep of the simulation correspond to figure 1. Compared with figure 1, this figure shows the blinker patterns develop stably. In fact, only two colours (black and grey) appear after timestep 1 and the value of the grey sites means unity.

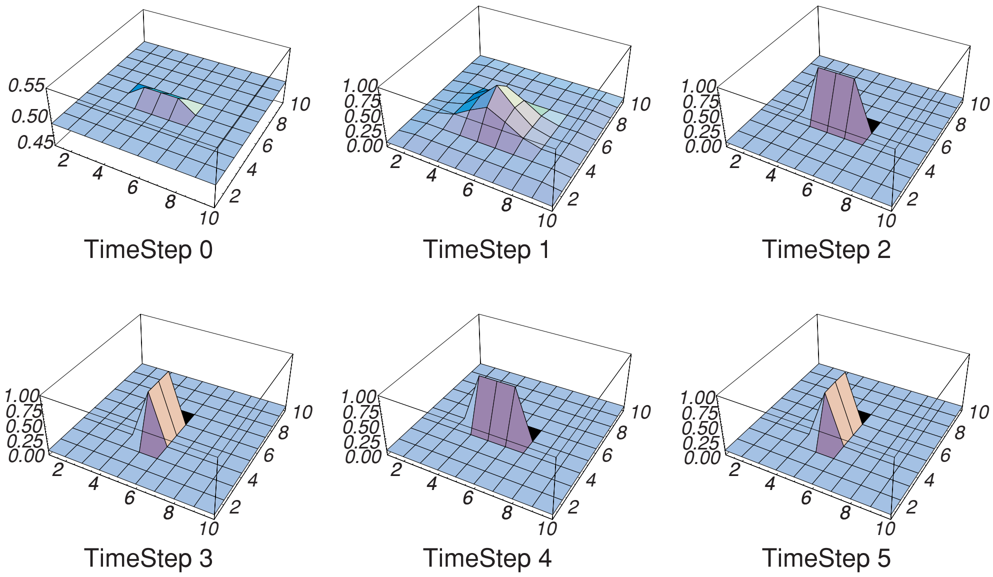

Figure 4. The result of the numerical calculation (a blinker). The result of the numerical calculation of the partial difference equation (27) by using the filtration functions (24) and (25), with $\varepsilon = 0.009$. The slightly different initial values 0.49 and 0.51 are given.

(This figure is in colour only in the electronic version)

Figure 4 shows the result of the numerical calculation of the partial difference equation (27) by using the the filtration functions (24) and (25) with $\varepsilon = 0.009$. This figure shows that the slightly different initial values 0.49 and 0.51 converge to almost 0.00 and 1.00 respectively, and the blinker pattern develops stably. This implies the filtration functions can recover the patterns of CA which are slightly left at the first timestep.

Figure 5 also shows the result of the numerical calculation of the partial difference equation (27) by using the the filtration functions (24) and (25) with $\varepsilon = 0.009$. In this figure, the initial values are given randomly from zero to one. (A white site means the value of it is 1.0. A black site has the value 0.0. The grey levels correspond to the magnitude of the value from zero to one.) However, in this case, the values of some sites which induce a typical pattern of the game of life—the glider—are accidentally greater than 0.5 at the initial timestep. Then the glider pattern of the game of life gradually appears from the initial values distributed almost randomly and evolves stably. This shows the filtration functions select the patterns of the game of life among the initial values and make them converge to the regular patterns.

These results show the filtration technique is a reliable method which provides the stable difference equations uniquely associated with the complicated CA. The procedure of inverse ultradiscretization is not always able to give a way to construct the stable difference equation uniquely. Besides, it is not known which limiting procedure of ultradiscretization is best [12]. It is practically hard to find the stable difference equation, particularly, in the cases of the higher dimensional CA and the complicated CA rules. For the filtration technique, the point is only one condition (26). We can construct the stable difference equation easily and automatically by the technique. However, the concrete difference equation (27) is made up of 140 terms by all combinations of $X_0(x)$ and $X_1(x)$. It is not easy to relate the equation to the continuous dynamical systems. It is obscure which limiting procedure is better to connect the constructed difference equations with the continuous systems. In the next section, we will

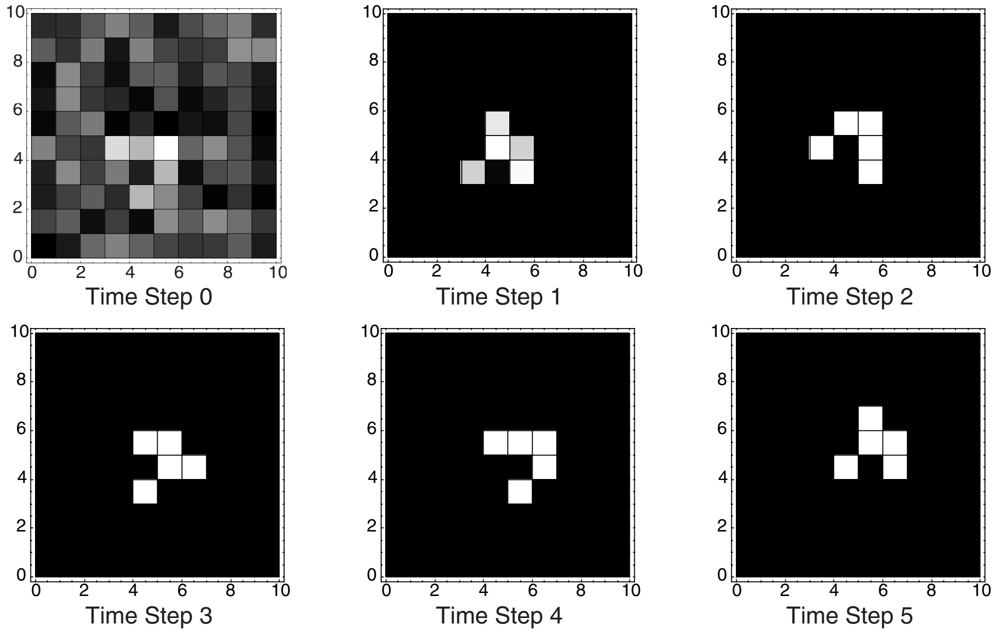

Figure 5. The result of the numerical calculation (a glider). The result of the numerical calculation of the partial difference equation (27) by using the the filtration functions (24) and (25), with $\varepsilon = 0.009$. The initial values are given randomly from zero to one. (A white site means the value of it is 1.0. A black site has the value 0.0. The grey levels correspond to the magnitude of the value from zero to one.) However, in this case, the values of some sites which induce a typical pattern of the game of life—a glider—are accidentally greater than 0.5 at the initial timestep.

show an example of the derivation of the continuous dynamical equation through the filtration technique.

4. Examination for the continuous systems

In this section, we apply the procedure obtained in the previous section to an ECA, in order to examine the connection between CA and continuous systems. In the paper [14], the relation between the Burgers equation and one of the ECA (Rule 184 CA) was clarified via the discrete Burgers equation and the ultradiscrete Burgers equation which was derived from the discrete Burgers equation with ultradiscretizaton. Referring to this result, we apply the procedure of the filtration functions to Rule 184 CA.

In the case of ECA, we have the general difference equation as

$$U_i^{t+1} = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} F_{\mathrm{ECA}}(i_1, i_2, i_3)\, X_{i_1}\big(U_{i-1}^t\big)\, X_{i_2}\big(U_i^t\big)\, X_{i_3}\big(U_{i+1}^t\big) \tag{28}$$

where $U_i^t \in \mathbb{R}$ is the value of the ith site ($i \in \mathbb{Z}$) and F_{ECA} is a map from $\{0, 1\} \times \{0, 1\} \times \{0, 1\}$ to $\{0, 1\}$. The rule of ECA is generally expressed as

$$v_i^{t+1} = F_{\mathrm{ECA}}\big(v_{i-1}^t, v_i^t, v_{i+1}^t\big) \tag{29}$$

where $v_i^t \in \{0, 1\}$ is the value of the ith site ($i \in \mathbb{Z}$). Here, we consider Rule 184 CA whose time evolution rule is given by

$$\frac{U_{j-1}^t U_j^t U_{j+1}^t}{U_j^{t+1}} = \frac{000}{0}, \frac{001}{0}, \frac{010}{0}, \frac{011}{1}, \frac{100}{1}, \frac{101}{1}, \frac{110}{0}, \frac{111}{1}.$$

From equation (28), we obtain the difference equation as

$$\begin{aligned} U_i^{t+1} = {} & X_0\big(U_{i-1}^t\big)\, X_1\big(U_i^t\big)\, X_1\big(U_{i+1}^t\big) + X_1\big(U_{i-1}^t\big)\, X_0\big(U_i^t\big)\, X_0\big(U_{i+1}^t\big) \\ & + X_1\big(U_{i-1}^t\big)\, X_0\big(U_i^t\big)\, X_1\big(U_{i+1}^t\big) + X_1\big(U_{i-1}^t\big)\, X_1\big(U_i^t\big)\, X_1\big(U_{i+1}^t\big). \end{aligned} \tag{30}$$

When the filtration functions $X_0(x)$ and $X_1(x)$ which satisfy the condition (26) are given by

$$X_0(x) := \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{1-2x}{2\varepsilon}\right)\right) \tag{31}$$

$$X_1(x) := \frac{1}{2}\left(1 - \frac{2}{\pi}\arctan\left(\frac{1-2x}{2\varepsilon}\right)\right) \tag{32}$$

where ε is a constant, by using $U_j^t \simeq X_1\big(U_j^t\big)$, we get the difference equation associated with Rule 184 CA as

$$\begin{aligned} U_j^{t+1} - U_j^t = {} & \frac{1}{\pi^2}\left(\arctan\left(\frac{1-2U_{j+1}^t}{2\varepsilon}\right) - \arctan\left(\frac{1-2U_{j-1}^t}{2\varepsilon}\right)\right)\arctan\left(\frac{1-2U_j^t}{2\varepsilon}\right) \\ & - \frac{1}{2\pi}\left(\arctan\left(\frac{1-2U_{j-1}^t}{2\varepsilon}\right) - 2\arctan\left(\frac{1-2U_j^t}{2\varepsilon}\right) + \arctan\left(\frac{1-2U_{j+1}^t}{2\varepsilon}\right)\right). \end{aligned} \tag{33}$$

In equation (33), we transform U_j^t to V_j^t by

$$V_j^t \equiv -\frac{1-2U_j^t}{2\varepsilon} \tag{34}$$

and make the parameter ε sufficiently large with the continuous limit of space and time as $V_j^{t'} \equiv \tilde{V}(j\Delta x, t'\Delta t)$ and $V(x,t) := \lim_{\Delta x, \Delta t \to +0} \tilde{V}(j\Delta x, t'\Delta t)$, $(x = j\Delta x, t = t'\Delta t)$. Note that changing the parameter ε for a larger presents the inverse process of discretizing the dependent variable. Then, we have

$$V_t = \frac{1}{(\pi\varepsilon)^2}\frac{\Delta x}{\Delta t} V V_x + \frac{1}{2\pi\varepsilon}\frac{\Delta x^2}{\Delta t} V_{xx} + \mathcal{O}(\varepsilon^{-3}, \Delta t, \Delta x^3). \tag{35}$$

When we regard $1/\varepsilon \sim \Delta x$ and $\Delta x^3/\Delta t \sim 1$, and neglect the higher order terms we get the Burgers equation:

$$V_t = \alpha V V_x + \beta V_{xx} \tag{36}$$

where $\alpha \equiv \frac{\Delta x}{(\pi\varepsilon)^2\Delta t}$ and $\beta \equiv \frac{\Delta x^2}{2\pi\varepsilon\Delta t}$. This result shows the Burgers equation is derived from Rule 184 CA by the filtration technique. Furthermore, this derivation does not depend on the choice of the filtration functions (recall the condition (26)). Therefore, it can be concluded that the filtration technique relates Rule 184 CA to the Burgers equation. This affords one example which implies that the filtration technique can give a way to the continuous dynamical systems from the CA rule while preserving the behaviour of the solutions.

In the above process, we should notice that the limiting procedure of obtaining the continuous equation (36) from the constructed difference equation (33) is followed without any difficulty. It is the crucial point for relating the CA rule to the partial differential equation. Actually, in the case that the difference equation concretely constructed from the general difference equation, such as (27) or (28), is complicated, it may be difficult to find the appropriate procedure to obtain the continuous equation. (As mentioned in the section 3, the two spatial dimensional difference equation is made up of 140 terms composed by all combinations of the filtration functions.) However, this fact will not mean that the filtration technique is useless, because it is the method which provides the difference equation associated with the CA. We can construct the stable difference equation associated with any other ECA as Rule 18 CA and Rule 90 CA which have a solution of the Sierpinski gasket type. Besides, it can be extended to more-than-two-state CA [15]. These facts are encouraging to try to challenge the further work, which is, for example, to investigate which limiting procedure is appropriate to obtaining the continuous equation from the difference equation, and to clarify which types of partial differential equations are related with CA rules. We believe that one approach to accumulating the evidence is finding other examples of correspondence by referring to results given by studies from different viewpoints such as [1, 2].

5. Concluding remarks

Following the proposal of [12], we concretely construct the partial difference equations, which preserve any time evolution patterns of the CA stably by the filtration technique. We illustrate how to develop a method of filtration applicable to the game of life, and verify that the filtration method provides the stable difference equation associated with the CA, compared with the inverse ultradiscretization. We also show the derivation of the Burgers equation from Rule 184 CA via the discrete Burgers equation constructed by the filtration method. We comment on the possibility that the filtration technique can contribute towards leading one to the continuous equation from the CA rule.

From the above results, we conclude that the filtration technique is a reliable method which works better than the inverse ultradiscretization at the point that the stable difference equation is constructed uniquely from the CA rule. One can automatically obtain the stable difference equation associated with the CA by following the concrete process shown in this paper. We believe that, considering the relation between the CA rules and the continuous systems, we can get a hint from the filtration technique. Moreover, the filtration technique may be applied to other subjects. We hope that much of the material presented here can be useful for further investigation in related (more difficult) problems.

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